Definition

The union of all faces of Δ^n is the **BOUNDARY** of Δ^n, written $\partial\Delta^n$.

The **OPEN SIMPLEX** $\mathring{\Delta}^n$ is $\Delta^n - \partial\Delta^n$, the interior of Δ^n.

Gluing simplices together in a principled way yields a **Δ-COMPLEX**.

Definition

A **Δ-COMPLEX STRUCTURE on a space X** is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:

(i) The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$

(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.

(iii) A set $A \subset X$ is open $\Longleftrightarrow$ $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

rules out trivialities like regarding all the points as individual vertices

One consequence of (iii) is that X can be built as a quotient space of a collection of disjoint simplices inductively: start with a discrete set of vertices, attach edges to produce graphs, attach triangles or 2-simplices, etc.

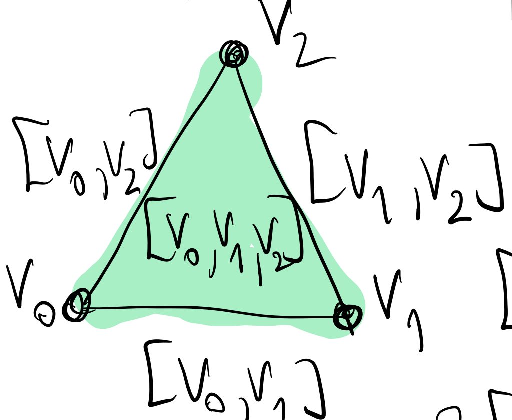
SIMPLICIAL HOMOLOGY

Definition

Let $\Delta_n(X)$ be the **free abelian group with basis the open n-simplices e^n_α of X.**

Elements of $\Delta_n(X)$ are called **n-chains** and can be written as finite formal sums $\sum_\alpha n_\alpha e_\alpha^n$ with coefficients $n_\alpha \in \mathbb{Z}$. Equivalently, we could write $\sum_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of e_α^n, with image the closure of e_α^n.

Example



$[V_0,V_1]+[V_1,V_2]+[V_0,V_2]$ is a 1-chain.

$[V_0,V_1,V_2]$ is a 2-chain

$2[V_0,V_1,V_2]$ is a 2-chain

the boundary of this 2-simplex consists of 1-simplices $[v_0, v_1], [v_1, v_2], [v_0, v_2]$

We might say that the boundary of f is the 1-chain formed by the sum of the faces $[v_0, v_1], [v_1, v_2], [v_0, v_2]$. However, we must take orientations into account.

Definition

the **BOUNDARY HOMOMORPHISM**

$$\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$$

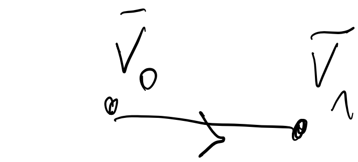
is specified on basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha |_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}.$$

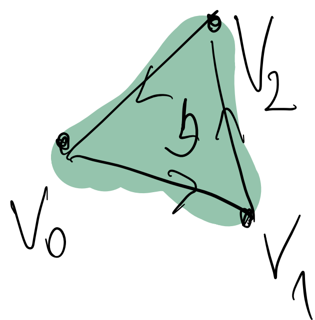
this is well-defined since each restriction $\sigma_\alpha |_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$

Examples

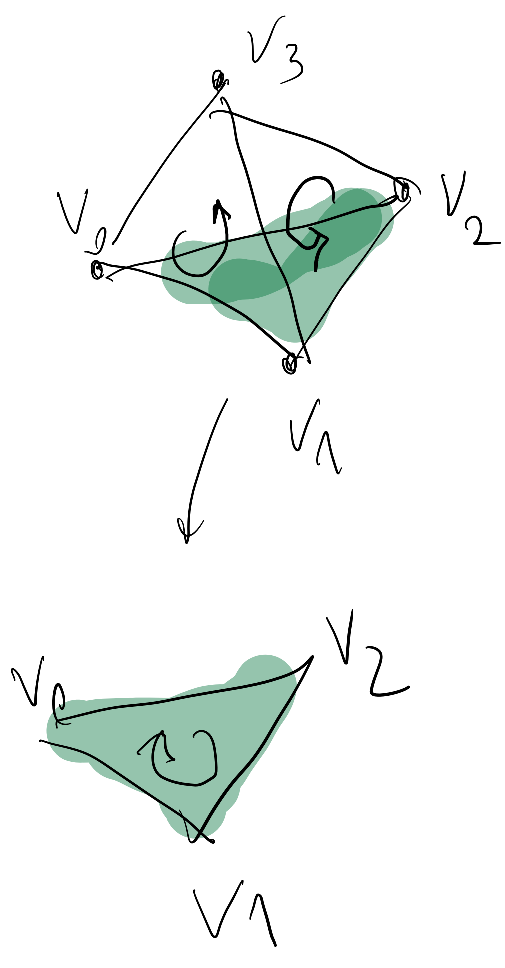
is the characteristic map of an $(n-1)$-simplex



$$\partial_1[v_0,v_1] = [v_1]-[v_0]$$



$$\partial_2[v_0,v_1,v_2] = [v_1,v_2]-[v_0,v_2]+[v_0,v_1]$$



$$\partial_3[v_0,v_1,v_2,v_3] = [v_1,v_2,v_3] - [v_0,v_2,v_3] + [v_0,v_1,v_3] - [v_0,v_1,v_2]$$

$$[v_0,v_2,v_1] = -[v_0,v_1,v_2]$$

Lemma

The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero.

Proof

$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$, and hence

$$\partial_{n-1}\partial_n(\sigma) = \partial_{n-1}\Big(\sum_i (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}\Big)$$

$$= \sum_i (-1)^i \, \partial_{n-1} \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} =$$

$$= \sum_i (-1)^i \Big(\sum_{j<i} (-1)^j \sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]}$$

$$+ \sum_{j>i} (-1)^{j-1} \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]} \Big)$$

$$= \sum_i \sum_{j<i} (-1)^{i+j} \sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]}$$

$$+ \sum_i \sum_{j>i} (-1)^{i+j-1} \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]}$$

The latter two summations cancel

since after switching i & j in the second sum, it becomes the negative of the first.

$\blacksquare$

The algebraic situation we have now is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \rightarrow \cdots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \circ \partial_{n+1} = 0$ for each n.

Such a sequence is called a **CHAIN COMPLEX**. NOTATION $(C_\bullet, \partial_\bullet)$

The equation $\partial_n \partial_{n+1} = 0$ is equivalent to the inclusion $\mathrm{Im}\, \partial_{n+1} \subset \mathrm{Ker}\, \partial_n$, where

$\operatorname{Im} \partial_{n+1}$ denotes the image of ∂_{n+1}

& $\operatorname{Ker} \partial_n$ the kernel of ∂_n.

Definition

Let $(C_\bullet, \partial_\bullet)$ be a chain complex.

The **n-th homology group of $(C_\cdot, \partial_\cdot)$**

$$H_n = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$$

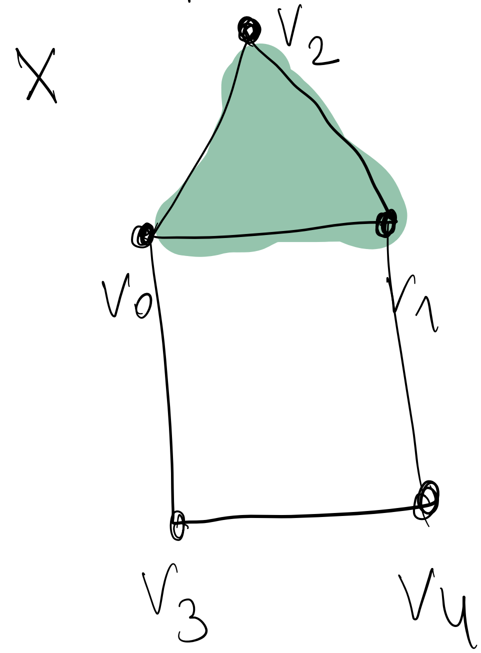
Elements of $\operatorname{Ker} \partial_n$ are called **cycles** and elements of $\operatorname{Im} \partial_{n+1}$ **boundaries**. Elements of H_n are cosets of $\operatorname{Im} \partial_{n+1}$, called **homology classes**. Two cycles representing the same homology class are said to be **homologous**.

Definition

When $C_n = \Delta_n(X)$, the homology group $\ker\partial_n / \operatorname{Im}\partial_{n+1}$ is denoted by $H_n^{\Delta}(X)$ and called the **nth simplicial homology group of X.**

Intuition: $H_n^{\Delta}(X)$ captures the information about n-dim holes in X.

Example 1 Homology groups of X



$\Delta_0(X) = \langle v_0, v_1, v_2, v_3, v_4 \rangle$

$\Delta_1(X) = \langle [v_0, v_2], [v_0, v_1], [v_0, v_3], [v_1, v_2], [v_1, v_4], [v_3, v_4] \rangle$

$\Delta_2(X) = \langle [v_0, v_1, v_2] \rangle$

$[v_1, v_2] - [v_0, v_2] + [v_0, v_1]$ $\Delta_3(X) = 0$

Chain complex associated to X

$$\to 0 \to \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{0} 0$$

$$\partial_2([v_0, v_1, v_2]) =$$

$$= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$\operatorname{Ker} \partial_2$ is trivial, so $H_2(X) = 0$.

Also $H_i(X) = 0$ for $i \geq 3$.